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The graphs induced by maximal totally isotropic flats of affine-unitary spaces

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ABSTRACT

Let $\delta = 0$ or 1, and let $AUG(2\nu + \delta, \mathbb{F}_q)$ be the $(2\nu + \delta)$-dimensional affine-unitary space over a finite field $\mathbb{F}_q$. Define a graph Γ whose vertex-set is the set of all maximal totally isotropic flats of $AUG(2\nu + \delta, \mathbb{F}_q)$, and in which F_1, F_2 are adjacent if and only if $\dim(F_1 \cup F_2) = \nu + 1$, for any $F_1, F_2 \in \Gamma$. We show that the distance between any two vertices in Γ is determined by means of dimension of their join and show that Γ is a vertex transitive graph with diameter ν and valency $(q^\nu - 1) + q^{3/2}\left[{\nu \atop 1}\right]_q$. We also show that any maximal clique in Γ can be changed under the group $AU_{2\nu+\delta}(\mathbb{F}_q)$ into the maximal clique Ω_1 with size $q(q^{1/2} + 1)$, the maximal clique Ω_3 with size $q^{\nu+\delta}$ ($\delta = 0$ or 1), or the maximal clique Ω_2 with size $q^{3/2} + 1$ ($\delta = 1$), and compute the number of maximal cliques containing a fixed vertex in Γ, and the total number of maximal cliques in Γ.

1. Introduction

In this section we shall first introduce the concepts of maximal totally isotropic flats in affine-unitary space, and then introduce our main results. We follow the notations and terminologies in [5–7].

Let $\mathbb{F}_q$ be a finite field with q elements, where $q = q_0^2$ and q_0 is a prime power. $\mathbb{F}_q$ has an involutive automorphism $a \mapsto \bar{a} = a^{q_0}$, whose fixed field is $\mathbb{F}_{q_0}$. Let $\delta = 0$ or 1, and let $\mathbb{F}_q^{(2\nu+\delta)}$ be the $(2\nu + \delta)$-dimensional row vector space over $\mathbb{F}_q$. Suppose that P is a subspace of $\mathbb{F}_q^{(2\nu+\delta)}$. We use the same letter P to denote the subspace as well its representation which is formed by a basis of P as rows.

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For $1 \leqslant i \leqslant 2\nu+\delta$, we use e_i to denote the $(2\nu+\delta)$-dimensional row vector whose ith component is 1 and other components are 0's. For any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q^{(2\nu+\delta)}$, denote by $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ the subspace of $\mathbb{F}_q^{(2\nu+\delta)}$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Denote by A^t the transpose of the matrix A.

Let

$$H_0 = \begin{pmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} H_0 & \\ & 1 \end{pmatrix}.$$

The *unitary group* of degree $2\nu+\delta$ over $\mathbb{F}_q$, denoted by $U_{2\nu+\delta}(\mathbb{F}_q)$, consists of all $(2\nu+\delta)\times(2\nu+\delta)$ matrix T over $\mathbb{F}_q$ satisfying $TH_\delta \bar{T}^t = H_\delta$ ($\delta = 0$ or 1), where $\bar{T}$ denotes the matrix obtained from T by replacing each entry in T by its image under the involutive automorphism $a \mapsto \bar{a}$. The vector space $\mathbb{F}_q^{(2\nu+\delta)}$ together with the right multiplication action of $U_{2\nu+\delta}(\mathbb{F}_q)$ is called the $(2\nu+\delta)$-dimensional *unitary space* over $\mathbb{F}_q$. An m-dimensional subspace P in $(2\nu+\delta)$-dimensional unitary space is said to be of *type* (m,r), if $PH_\delta \bar{P}^t$ is of rank r. In particular, subspaces of type $(m,0)$ are called m-dimensional *totally isotropic subspaces*, and ν-dimensional totally isotropic subspaces are called *maximal totally isotropic subspaces*. It is known that subspaces of type (m,r) exist if and only if $2r \leqslant 2m \leqslant 2\nu+\delta+r$.

Suppose P is a subspace of type (m,r) in $(2\nu+\delta)$-dimensional unitary space $\mathbb{F}_q^{(2\nu+\delta)}$. A coset of $\mathbb{F}_q^{(2\nu+\delta)}$ relative to a subspace P of type (m,r) is called a *(m,r)-flat*. The *dimension* of a (m,r)-flat $U+x$ is defined to be the dimension of the subspace U, denoted by $\dim(U+x)$.

A flat F_1 is said to be *incident* with a flat F_2, if F_1 contains or is contained in F_2. The point set $\mathbb{F}_q^{(2\nu+\delta)}$ with all the flats and the incidence relation among them defined above is said to be the $(2\nu+\delta)$-dimensional *affine-unitary space*, denoted by $AUG(2\nu+\delta, \mathbb{F}_q)$. Denote by $F_1 \cap F_2$ the intersection of the flats F_1 and F_2, and by $F_1 \cup F_2$ the minimum flat containing both F_1 and F_2.

The set of matrices of the form

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix},$$

where $T \in U_{2\nu+\delta}(\mathbb{F}_q)$ and $v \in \mathbb{F}_q^{(2\nu+\delta)}$, forms a group under matrix multiplication, which is denoted by $AU_{2\nu+\delta}(\mathbb{F}_q)$ and called the *affine-unitary group* of degree $2\nu+\delta$ over $\mathbb{F}_q$. Define the action of $AU_{2\nu+\delta}(\mathbb{F}_q)$ on $AUG(2\nu+\delta, \mathbb{F}_q)$ as follows:

$$AUG(2\nu+\delta, \mathbb{F}_q) \times AU_{2\nu+\delta}(\mathbb{F}_q) \to AUG(2\nu+\delta, \mathbb{F}_q),$$
$$\left(x, \begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}\right) \mapsto xT + v.$$

The above action induces an action on the set of flats of $AUG(2\nu+\delta, \mathbb{F}_q)$, i.e., a flat $P+x$ is carried by

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix} \in AU_{2\nu+\delta}(\mathbb{F}_q)$$

into the flat $PT + (xT+v)$. By [6] $AU_{2\nu+\delta}(\mathbb{F}_q)$ is transitive on the set of (m,r)-flats in $AUG(2\nu+\delta, \mathbb{F}_q)$ for a given (m,r).

Let m_1, m_2 be two integers. Then the *Gaussian coefficient* is

$$\begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q = \frac{\prod_{j=m_2-m_1+1}^{m_2}(q^j-1)}{\prod_{j=1}^{m_1}(q^j-1)}.$$

By convenience $\begin{bmatrix} m_2 \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} m_2 \\ m_1 \end{bmatrix}_q = 0$, whenever $m_1 < 0$ or $m_2 < m_1$.

Define a graph Γ whose vertex-set is the set of all maximal totally isotropic flats of $AUG(2\nu+\delta,\mathbb{F}_q)$, and in which F_1, F_2 are adjacent if and only if $\dim(F_1 \cup F_2) = \nu + 1$, for any $F_1, F_2 \in \Gamma$.

The vertex-set of Γ is written by $V(\Gamma)$, and $\partial(F_1, F_2)$ means the distance between vertices F_1 and F_2. For any $F_1 \in V(\Gamma)$, we use $\Gamma_r(F_1)$ to denote the set of vertices F_2 in Γ satisfying $\partial(F_1, F_2) = r$, and write $\Gamma(F_1) = \Gamma_1(F_1)$.

In [3], we studied the graphs induced by maximal totally isotropic flats of affine-symplectic spaces. In this paper, we obtain following results.

Theorem 1.1. *Let $1 \leqslant r \leqslant \nu$. Then $|V(\Gamma)| = q^{\nu+\delta}\prod_{i=1}^{\nu}(q^{i+\delta-1/2}+1)$, and Γ is a vertex transitive graph.*

Theorem 1.2. *For any two vertices F_1, F_2 of Γ, $\dim(F_1 \cup F_2) = \nu + r$ if and only if $\partial(F_1, F_2) = r$. In particular, Γ is a graph with diameter $\nu + \delta$.*

Theorem 1.3. *Let $1 \leqslant r \leqslant \nu + \delta$. Then for any vertex F_1 of Γ,*

$$\left|\Gamma_r(F_1)\right| = \left(q^{\nu+\delta-r+1} - 1\right)q^{(r-1)(r+1+2\delta)/2}\begin{bmatrix}\nu\\ r-1\end{bmatrix}_q + q^{r(r+2+2\delta)/2}\begin{bmatrix}\nu\\ r\end{bmatrix}_q.$$

In particular, Γ is a regular graph with valency $(q^{\nu+\delta}-1) + q^{(3+2\delta)/2}\left[\begin{smallmatrix}\nu\\ 1\end{smallmatrix}\right]_q$.

Theorem 1.4. *For a fixed maximal totally isotropic subspace $\langle e_1, e_2, \ldots, e_\nu\rangle$, any maximal clique in Γ can be changed under the group $AU_{2\nu+\delta}(\mathbb{F}_q)$ into*

$$\Omega_1 = \left\{F \subseteq \langle e_1, e_2, \ldots, e_{\nu+1}\rangle \;\middle|\; F \text{ is a maximal totally isotropic flat}\right\},$$

or

$$\Omega_2 = \left\{\langle e_2, \ldots, e_\nu\rangle \subseteq F \;\middle|\; F \text{ is a maximal totally isotropic subspace}\right\}, \quad \text{where } \delta = 1,$$

or

$$\Omega_3 = \left\{\langle e_1, e_2, \ldots, e_\nu\rangle + x \;\middle|\; x \in \mathbb{F}_q^{(2\nu+\delta)}\right\}.$$

The size of maximal cliques in Γ is $(q^{1/2}+1)q$, $q^{3/2}+1$ $(\delta = 1)$ or $q^{\nu+\delta}$, respectively.

Theorem 1.5. *For any $F \in V(\Gamma)$, there are precisely $q^\delta(q^\nu-1)/(q-1) + \delta q(q^\nu-1)/(q-1) + 1$ maximal cliques containing F in Γ. Furthermore, the total number of maximal cliques in Γ is equal to*

$$\frac{q^{\nu-1+2\delta}(q^\nu-1)\prod_{i=1}^{\nu}(q^{i+\delta-1/2}+1)}{(q-1)(q^{1/2}+1)} + \prod_{i=1}^{\nu}\left(q^{i+\delta-1/2}+1\right) + \delta\frac{q^{\nu+2}(q^\nu-1)\prod_{i=2}^{\nu}(q^{i+1/2}+1)}{(q-1)}.$$

Remarks. For a given $x \in \mathbb{F}_q^{(2\nu+\delta)}$, let $\mathcal{M}(x)$ be the set of all maximal totally isotropic flats containing x of $AUG(2\nu+\delta,\mathbb{F}_q)$. Let Γ' be a subgraph of Γ induced by $\mathcal{M}(x)$. Then Γ' is isomorphic to the Dual polar graphs ${}^2A_{2\nu-1}(q_0)$ or ${}^2A_{2\nu}(q_0)$ (see [1] and [2]).

2. Proof of main results

We begin with some useful results which are needed in the proof of the above theorems.

Proposition 2.1. *(See [4] and [5].) Let $F_1 = V_1 + x_1$ and $F_2 = V_2 + x_2$ be any two flats of $AUG(2\nu+\delta,\mathbb{F}_q)$, where V_1 and V_2 are two vector subspaces of $\mathbb{F}_q^{(2\nu+\delta)}$, and $x_1, x_2 \in \mathbb{F}_q^{(2\nu+\delta)}$. Then*

(i) $F_1 \cap F_2 \neq \emptyset$ *if and only if* $x_2 - x_1 \in V_1 + V_2$.
(ii) *If* $F_1 \cap F_2 \neq \emptyset$, *then* $F_1 \cap F_2 = V_1 \cap V_2 + x$, *where* $x \in F_1 \cap F_2$.
(iii) $F_1 \cup F_2 = V_1 + V_2 + \langle x_2 - x_1 \rangle + x_1$. *In particular,*

$$\dim(F_1 \cup F_2) = \begin{cases} \dim F_1 + \dim F_2 - \dim(F_1 \cap F_2), & \text{if } F_1 \cap F_2 \neq \emptyset, \\ \dim F_1 + \dim F_2 - \dim(V_1 \cap V_2) + 1, & \text{if } F_1 \cap F_2 = \emptyset. \end{cases}$$

For $0 \leqslant i \leqslant \nu$, let P_0 denote a maximal totally isotropic subspace of $\mathbb{F}_q^{(2\nu+\delta)}$. By [5], $U_{2\nu+\delta}(\mathbb{F}_q)$ acts transitively on the set of subspaces of same type and $\langle e_1, e_2, \ldots, e_\nu \rangle$ is a maximal totally isotropic subspace. So, we may assume that $P_0 = \langle e_1, e_2, \ldots, e_\nu \rangle$. Let $\mathcal{M}(i, P_0; 2\nu + \delta)$ denote the set of all maximal totally isotropic subspaces Q of $\mathbb{F}_q^{(2\nu+\delta)}$ satisfying $\dim(P_0 \cap Q) = i$, then we have the following lemma.

Lemma 2.2.

$$\left|\mathcal{M}(i, P_0; 2\nu + \delta)\right| = q^{\frac{(\nu-i)^2}{2} + (\nu-i)\delta} \begin{bmatrix} \nu \\ i \end{bmatrix}_q.$$

Proof. Let Q_0, Q_0' be any two i-dimensional subspaces of P_0. Then there is a

$$T = \begin{pmatrix} T_1 & & \\ & \bar{T}_1^{t^{-1}} & \\ & & I^{(\delta)} \end{pmatrix} \in U_{2\nu+\delta}(\mathbb{F}_q)$$

which fixes the subspace P_0 and carries Q_0 into Q_0', where T_1 is a $\nu \times \nu$ nonsingular matrix. Let $\mathcal{M}(Q_0, P_0; 2\nu + \delta)$ (resp. $\mathcal{M}(Q_0', P_0; 2\nu + \delta)$) denote the set of all maximal totally isotropic subspaces Q (resp. Q') of $\mathbb{F}_q^{(2\nu+\delta)}$ satisfing $P_0 \cap Q = Q_0$ (resp. $P_0 \cap Q' = Q_0'$). It is easy to see that T carries $\mathcal{M}(Q_0, P_0; 2\nu + \delta)$ to $\mathcal{M}(Q_0', P_0; 2\nu + \delta)$, hence we have $|\mathcal{M}(Q_0, P_0; 2\nu + \delta)| = |\mathcal{M}(Q_0', P_0; 2\nu + \delta)|$.

Let $Q_0 = (I^{(i)} 0^{(i, 2\nu+\delta-i)})$. For any element $Q \in \mathcal{M}(Q_0, P_0; 2\nu + \delta)$, since Q is totally isotropic, it has a matrix representation of the form

$$\begin{pmatrix} I^{(i)} & 0 & 0^{(i)} & 0 & 0^{(\nu,\delta)} \\ 0 & A_2 & 0 & A_4 & A_5 \end{pmatrix}, \tag{1}$$

where both A_2 and A_4 are $(\nu - i) \times (\nu - i)$ matrices, rank $(A_4 A_5) = \nu - i$ and

$$A_2 \bar{A}_4^t + A_4 \bar{A}_2^t + A_5 I^{(\delta)} \bar{A}_5^t = 0. \tag{2}$$

We claim that rank $A_4 = \nu - i$. If rank $A_4 < \nu - i$, then we can choose a matrix representation of Q as the form (1) such that all elements in the first row of A_4 are zero. If $\delta = 0$, then rank $(A_4 A_5) = \operatorname{rank} A_4 < \nu - i$, a contradiction. If $\delta = 1$, then we can deduce from (2) that the entry at the $(1,1)$ of A_5 is zero, so rank $(A_4 A_5) < \nu - i$, a contradiction. Without loss of generality, we may choose $A_4 = I^{(\nu-i)}$. Then $A_2 + \bar{A}_2^t + A_5 I^{(\delta)} \bar{A}_5^t = 0$. So

$$\left|\mathcal{M}(Q_0, P_0; 2\nu + \delta)\right| = q^{(\nu-i)^2/2 + (\nu-i)\delta}.$$

Since the matrix representation of Q as the form (1) is unique and there are $\left[\begin{smallmatrix} \nu \\ i \end{smallmatrix}\right]_q$ $Q_0's$ in P_0 by [4, Theorem 1.7], we obtain

$$\left|\mathcal{M}(i, P_0; 2\nu + \delta)\right| = \left|\mathcal{M}(Q_0, P_0; 2\nu + \delta)\right| \begin{bmatrix} \nu \\ i \end{bmatrix}_q = q^{(\nu-i)^2/2 + (\nu-i)\delta} \begin{bmatrix} \nu \\ i \end{bmatrix}_q. \qquad \square$$

Lemma 2.3. *Let F_1, F_2 be any two vertices of Γ with* $\dim(F_1 \cup F_2) = \nu + 1$. *Then F_1 and F_2 can be changed under $AU_{2\nu+\delta}(\mathbb{F}_q)$ into*

$$\langle e_1, \dots, e_\nu \rangle \quad \textit{and} \quad \langle e_2, \dots, e_{\nu+1} \rangle, \tag{3}$$

or

$$\langle e_1, \dots, e_\nu \rangle \quad \textit{and} \quad \langle e_1, \dots, e_\nu \rangle + e_{\nu+1}, \tag{4}$$

or

$$\langle e_1, \dots, e_\nu \rangle \quad \textit{and} \quad \langle e_1, \dots, e_\nu \rangle + \lambda e_{2\nu+1}, \quad \textit{where } \delta = 1 \textit{ and } \lambda \in \mathbb{F}_q^*, \tag{5}$$

simultaneously.

Proof. Let $F_1 = V + x$ and $F_2 = W + y$, where both V and W are maximal totally isotropic subspaces of $\mathbb{F}_q^{(2\nu+\delta)}$ and $x, y \in \mathbb{F}_q^{(2\nu+\delta)}$. We distinguish the following two cases:

Case 1. $F_1 \cap F_2 \neq \emptyset$. By Proposition 2.1 we deduce that $\dim(F_1 \cap F_2) = \nu - 1$. Choose the matrix representation of the subspaces V, W and $U + W$ as follows:

$$V = \begin{pmatrix} V_1 \\ D \end{pmatrix}, \qquad W = \begin{pmatrix} D \\ W_1 \end{pmatrix} \quad \text{and} \quad V + W = \begin{pmatrix} V_1 \\ D \\ W_1 \end{pmatrix},$$

where both V_1 and W_1 are $1 \times 2\nu$ matrices, D is a $(\nu - 1) \times 2\nu$ matrix. Since $V + W$ is a subspace of type $(\nu + 1, 2)$, there exists $T \in U_{2\nu+\delta}(\mathbb{F}_q)$ such that

$$\begin{pmatrix} V_1 \\ D \\ W_1 \end{pmatrix} T = \left(I^{(\nu+1)} 0^{(\nu+1, \nu+\delta-1)} \right).$$

Clearly,

$$\begin{pmatrix} T & 0 \\ -xT & 1 \end{pmatrix} \in AU_{2\nu+\delta}(\mathbb{F}_q)$$

carries F_1 and F_2 into

$$\langle e_1, \dots, e_\nu \rangle \quad \text{and} \quad \langle e_2, \dots, e_{\nu+1} \rangle + (y - x)T, \tag{6}$$

respectively. By $F_1 \cap F_2 \neq \emptyset$, we have $\langle e_1, \dots, e_\nu \rangle \cap (\langle e_2, \dots, e_{\nu+1} \rangle + (y - x)T) \neq \emptyset$. So $(y - x)T$ can be taken as the following form:

$$(y - x)T = (x_1, \dots, x_\nu, 0, \dots, 0).$$

Then the element

$$\begin{pmatrix} I & 0 \\ (x - y)T & 1 \end{pmatrix} \in AU_{2\nu+\delta}(\mathbb{F}_q)$$

carries two flats in (6) into that listed in (3).

Case 2. $F_1 \cap F_2 = \emptyset$. By Proposition 2.1 we deduce that $V = W$ and $y - x \notin V$. Note that

$$\begin{pmatrix} V \\ y-x \end{pmatrix} H_\delta \overline{\begin{pmatrix} V \\ y-x \end{pmatrix}}^t = \begin{pmatrix} 0 & V H_\delta \overline{(y-x)}^t \\ (y-x) H_\delta \bar{V}^t & (y-x) H_\delta \overline{(y-x)}^t \end{pmatrix}.$$

If $V H_\delta \overline{(y-x)}^t \neq 0$, then we can suppose that

$$\begin{pmatrix} V \\ y-x \end{pmatrix} H_\delta \overline{\begin{pmatrix} V \\ y-x \end{pmatrix}}^t = \begin{pmatrix} 0 & \tilde{e}_1^t \\ \tilde{e}_1 & (y-x) H_\delta \overline{(y-x)}^t \end{pmatrix},$$

where $\tilde{e}_1$ is the ν-dimensional row vector whose 1th entry is 1 and all other entries are zeros. So by [5, Lemma 5.1], there exists $\mu \in \mathbb{F}_q$ such that

$$\begin{pmatrix} V \\ \mu e_1 V + y - x \end{pmatrix} H_\delta \overline{\begin{pmatrix} V \\ \mu e_1 V + y - x \end{pmatrix}}^t = \begin{pmatrix} 0 & \tilde{e}_1^t \\ \tilde{e}_1 & 0 \end{pmatrix}.$$

It follows that there exists $T \in U_{2\nu+\delta}(\mathbb{F}_q)$ such that $VT = \langle e_1, \ldots, e_\nu \rangle$ and $(\mu e_1 V + y - x)T = e_{\nu+1}$. Clearly,

$$\begin{pmatrix} T & 0 \\ -xT & 1 \end{pmatrix} \in AU_{2\nu+\delta}(\mathbb{F}_q)$$

carries F_1 and F_2 into that listed in (4), respectively.

If $V H_\delta \overline{(y-x)}^t = 0$, then $\delta = 1$. By [5, Lemma 5.1], there exists $\lambda \in \mathbb{F}_q^*$ such that

$$\begin{pmatrix} V \\ \lambda^{-1}(y-x) \end{pmatrix} H_\delta \overline{\begin{pmatrix} V \\ \lambda^{-1}(y-x) \end{pmatrix}}^t = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So there exists $T \in U_{2\nu+1}(\mathbb{F}_q)$ such that $VT = \langle e_1, \ldots, e_\nu \rangle$ and $\lambda^{-1}(y-x)T = e_{2\nu+1}$. Clearly,

$$\begin{pmatrix} T & 0 \\ -xT & 1 \end{pmatrix} \in AU_{2\nu+1}(\mathbb{F}_q)$$

carries F_1 and F_2 into that listed in (5), respectively. $\square$

Lemma 2.4. *Let $T \in AU_{2\nu+\delta}(\mathbb{F}_q)$ and $\sigma_T : V(\Gamma) \to V(\Gamma)$, $F \mapsto FT$. Then $\sigma_T \in \mathrm{Aut}(\Gamma)$.*

Proof. Since T is nonsingular, σ_T is a bijection. For any $F_1, F_2 \in V(\Gamma)$, $\dim(F_1 \cup F_2) = \nu + 1$ if and only if $\dim(F_1 T \cup F_2 T) = \nu + 1$. It follows that F_1 and F_2 are adjacent if and only if $F_1 T$ and $F_2 T$ are adjacent. Hence $\sigma_T \in \mathrm{Aut}(\Gamma)$. $\square$

Proof of Theorem 1.1. By [5, Corollary 5.20], the number of maximal totally isotropic subspaces of the unitary space $\mathbb{F}_q^{(2\nu+\delta)}$ is equal to $\prod_{i=1}^{\nu}(q^{i+\delta-1/2} + 1)$. So

$$|V(\Gamma)| = q^{\nu+\delta} \prod_{i=1}^{\nu} (q^{i+\delta-1/2} + 1).$$

By Lemma 2.4, Γ is vertex transitivity. $\square$

Proof of Theorem 1.2. We prove by induction on r. The case $r=1$ is trivial. Suppose that for any two vertices F_1, F_2 of Γ, $\dim(F_1 \cup F_2) = \nu + i$ if and only if $\partial(F_1, F_2) = i$, where $1 \leqslant i < r$.

Let $\dim(F_1 \cup F_2) = \nu + r$. Write $F_1 = U + x$ and $F_2 = V + y$. By Proposition 2.1, $\dim(U + V + \langle y - x\rangle) = \nu + r$. We distinguish the following two cases:

Case 1. $y - x \in U + V$. Then by Proposition 2.1 $F_1 \cap F_2 \neq \emptyset$. So we can suppose that $U + x = U + w$ and $V + y = V + w$, where $w \in F_1 \cap F_2$. Then $\dim(U \cap V) = \dim(F_1 \cap F_2) = \nu - r$. Since both $U + V$ and $\langle e_1, e_2, \ldots, e_{\nu+r}\rangle$ are subspaces of type $(\nu + r, 2r)$, there exists a $T \in U_{2\nu+\delta}(\mathbb{F}_q)$ such that $UT = \langle e_1, \ldots, e_\nu\rangle$ and $VT = \langle e_{r+1}, \ldots, e_{r+\nu}\rangle$. Let $W = \langle e_2, \ldots, e_{\nu+1}\rangle \in V(\Gamma)$ and $F = WT^{-1} + w$, then $\dim(F_1 \cup F) = \dim(WT^{-1} + U) = \dim(W + UT) = \nu + 1$ and $\dim(F_2 \cup F) = \dim(WT^{-1} + V) = \dim(W + VT) = \nu + r - 1$. The induction hypothesis applied to $\dim(F_2 \cup F) = \nu + r - 1$ implies $\partial(F_2, F) = r - 1$. Hence $\dim(F_1 \cup F_2) = \nu + r$.

Conversely, let $\partial(F_1, F_2) = r$. Then there exists $F \in V(\Gamma)$ such that $\partial(F_1, F) = r - 1$ and $\partial(F, F_2) = 1$. By the induction hypothesis, $\dim(F_1 \cup F) = \nu + r - 1$ and $\dim(F \cup F_2) = \nu + 1$. By Proposition 2.1 and $F \subseteq (F_1 \cup F) \cap (F \cup F_2)$, we obtain that

$$\begin{aligned}\dim\big((F_1 \cup F) \cup (F \cup F_2)\big) &= \dim(F_1 \cup F) + \dim(F \cup F_2) - \dim\big((F_1 \cup F) \cap (F \cup F_2)\big)\\ &\leqslant 2\nu + r - \dim F\\ &= \nu + r.\end{aligned}$$

Note that $F_1 \cup F_2 \subseteq (F_1 \cup F) \cup (F \cup F_2)$. It follows that $\dim(F_1 \cup F_2) \leqslant \nu + r$. If $\dim(F_1 \cup F_2) = \nu + t$ $(t < r)$, then by the induction hypothesis, $\partial(F_1, F_2) = t < r$, a contradiction. Therefore $\partial(F_1, F_2) = r$.

Case 2. $y - x \notin U + V$. Then by Proposition 2.1 $F_1 \cap F_2 = \emptyset$. Since $U + V$ is a subspace of type $(\nu + r - 1, 2r - 2)$, there exists a $T \in U_{2\nu+\delta}(\mathbb{F}_q)$ such that $UT = \langle e_1, \ldots, e_\nu\rangle$ and $VT = \langle e_r, \ldots, e_{r+\nu-1}\rangle$. Let $W = \langle e_2, \ldots, e_{\nu+1}\rangle \in V(\Gamma)$ and $F = WT^{-1} + x$, then $\dim(F_1 \cup F) = \dim(WT^{-1} + U) = \dim(W + UT) = \nu + 1$. Since $W + VT \subseteq UT + VT$, $WT^{-1} + V = (W + VT)T^{-1}$ and $UT + VT = (U + V)T$, we obtain $WT^{-1} + V = (W + VT)T^{-1} \subseteq (UT + VT)T^{-1} = U + V$. So $y - x \notin WT^{-1} + V$ by $y - x \notin U + V$. By $\dim(WT^{-1} + V) = \dim(W + VT) = \nu + r - 2$, $\dim(F_2 \cup F) = \dim(WT^{-1} + V) + 1 = \nu + r - 1$. The induction hypothesis applied to $\dim(F_2 \cup F) = \nu + r - 1$ implies $\partial(F_2, F) = r - 1$. Hence $\partial(F_1, F_2) = r$.

Conversely, the proof is similar to that of case 1, and will be omitted. □

Proof of Theorem 1.3. Let F be any vertex of Γ. Since the group $AU_{2\nu+\delta}(\mathbb{F}_q)$ acts transitively on the set of all vertices of Γ, we can assume that $F = \langle e_1, \ldots, e_\nu\rangle$.

Let $F' = U + x$ be any element of $\Gamma_r(F)$. Then $\dim(F \cup F') = \nu + r$, i.e., $\dim(F + U + \langle x\rangle) = \nu + r$. By Proposition 2.1, we have $\dim(F \cap U) = \nu - r + 1$ and $x \notin F + U$, or $\dim(F \cap U) = \nu - r$ and $x \in F + U$. We distinguish the following two cases:

Case 1. $\dim(F \cap U) = \nu - r + 1$ and $x \notin F + U$. Then $F + U$ is a subspace of type $(\nu + r - 1, 2(r - 1))$. By Lemma 2.2, the number of maximal totally isotropic subspaces U of $\mathbb{F}_q^{(2\nu+\delta)}$ satisfying $\dim(F \cap U) = \nu - r + 1$ is equal to $q^{(r-1)^2/2+(r-1)\delta}\left[{\nu \atop \nu-r+1}\right]_q$. For a given U, by Proposition 2.1, there exist q^{r-1} maximal totally isotropic flats $U + x$ such that $(U + x) \cap F \neq \emptyset$. It follows that there exist $q^{\nu+\delta} - q^{r-1}$ maximal totally isotropic flats $U + x$ such that $(U + x) \cap F = \emptyset$. Therefore, we obtain there are altogether $(q^{\nu+\delta} - q^{r-1})q^{(r-1)^2/2+(r-1)\delta}\left[{\nu \atop \nu-r+1}\right]_q$ elements of $\Gamma_r(F)$.

Case 2. $\dim(F \cap U) = \nu - r$ and $x \in F + U$. Then $F + U$ is a subspace of type $(\nu + r, 2r)$. Similar to the proof of case 1, we obtain that there are altogether $q^r q^{r^2/2+r\delta}\left[{\nu \atop \nu-r}\right]_q$ elements of $\Gamma_r(F)$, corresponding to this case.

So

$$\big|\Gamma_r(F_1)\big| = \big(q^{\nu+\delta-r+1} - 1\big)q^{(r-1)(r+1+2\delta)/2}\begin{bmatrix}\nu\\ r-1\end{bmatrix}_q + q^{r(r+2+2\delta)/2}\begin{bmatrix}\nu\\ r\end{bmatrix}_q.$$

In particular, Γ is a regular graph with valency $(q^{\nu+\delta} - 1) + q^{(3+2\delta)/2}\left[{\nu \atop 1}\right]_q$. □

Proof of Theorem 1.4. By Lemma 2.3, we know that any two adjacent vertices of Γ can be changed under $AU_{2\nu+\delta}(\mathbb{F}_q)$ into (3), (4) or (5). In order to determine neighbors of two adjacent vertices it suffices to consider the adjacent vertices listed in (3), (4) or (5), respectively.

First, suppose that F_1 and F_2 are as listed in (3). For any $F = V + x \in \Gamma(F_1) \cap \Gamma(F_2)$, we have $V = F_1$ and $x \notin F_1$, or $V \neq F_1$ and $x \in F_1 + V$ by Proposition 2.1. We distinguish the following two cases:

Case 1. $V = F_1$ and $x \notin F_1$. Then $F \neq F_2$ and $x \in V + F_2$ by Proposition 2.1. So $F = V + ae_{\nu+1} = \langle e_1, \ldots, e_\nu \rangle + ae_{\nu+1}$, where $0 \neq a \in \mathbb{F}_q$.

Case 2. $V \neq F_1$ and $x \in F_1 + V$. Then $\dim(F_1 + V) = \nu + 1$. We further distinguish the following two cases:

Case 2.1. $V = F_2$ and $x \notin F_2$. Then $F = V + ae_1 = \langle e_2, \ldots, e_{\nu+1} \rangle + ae_1$, where $0 \neq a \in \mathbb{F}_q$.

Case 2.2. $V \neq F_2$ and $x \in V + F_2$. Then $\dim(F_2 + V) = \nu + 1$ and $x \in (F_1 + V) \cap (V + F_2)$. Clearly, $\nu - 1 = \dim(F_1 \cap F_2) \geqslant \dim(F_1 \cap F_2 \cap V) = \dim(F_1 \cap V) + \dim F_2 - \dim((F_1 \cap V) + F_2) \geqslant 2\nu - 1 - \dim(F_1 + F_2) = \nu - 2$. Now we will prove $\langle e_2, \ldots, e_\nu \rangle \subseteq V$. If $\dim(F_1 \cap F_2 \cap V) = \nu - 2$. Then V has a matrix representation of form

$$V = \begin{array}{cc} \begin{array}{ccccc} 1 & \nu-1 & 1 & \nu-1 & \delta \end{array} & \\ \begin{pmatrix} 0 & A & 0 & 0 & 0 \\ 1 & B & 0 & 0 & 0 \\ 0 & C & D & E & F \end{pmatrix} & \begin{array}{c} \nu-2 \\ 1 \\ 1 \end{array} \end{array},$$

where $\operatorname{rank} A = \nu - 2$, $A\bar{E}^t = 0$, $B\bar{E}^t + \bar{D}^t = 0$, $C\bar{E}^t + E\bar{C}^t + F\bar{F}^t = 0$. If $E \neq 0$, then $\dim(F_2 \cap V) = \nu - 2$, a contradiction. If $E = 0$, then $D = 0$, $F = 0$. So $V = F_1$, a contradiction. Therefore, $\langle e_2, \ldots, e_\nu \rangle \subseteq V$ and $\dim(F_1 \cap F_2 \cap V) = F_1 \cap F_2$. So V has a matrix representation of the form

$$V = \begin{array}{cc} \begin{array}{ccccc} 1 & \nu-1 & 1 & \nu-1 & \delta \end{array} & \\ \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ B & 0 & C & 0 & E \end{pmatrix} & \begin{array}{c} \nu-1 \\ 1 \end{array} \end{array}, \tag{7}$$

where $B\bar{C}^t + C\bar{B}^t + EI^{(\delta)}\bar{E}^t = 0$. If $C = 0$, then $E = 0$. So $V = F_1$, a contradiction. If $C \neq 0$, then we can suppose that $C = 1$. So the number subspaces as form (7) is equal to $q^{\delta+1/2} - 1$. We further distinguish the following two cases:

Case 2.2.1. $V \subseteq \langle e_1, \ldots, e_{\nu+1} \rangle$. Then $\langle e_1, \ldots, e_\nu \rangle + V = \langle e_2, \ldots, e_{\nu+1} \rangle + V = \langle e_1, \ldots, e_{\nu+1} \rangle$. So $x \in (F_1 + V) \cap (V + F_2) = \langle e_1, \ldots, e_{\nu+1} \rangle$. By Theorem 1.2 and all cases above,

$$\Omega_1 = \{ F \subseteq \langle e_1, e_2, \ldots, e_{\nu+1} \rangle \mid F \text{ is a maximal totally isotropic flat} \}$$

is a clique containing F_1 and F_2.

Case 2.2.2. $V \nsubseteq \langle e_1, \ldots, e_{\nu+1} \rangle$. Then $\delta = 1$ and $E \neq 0$. Since $\dim(V + \langle e_1, \ldots, e_{\nu+1} \rangle) = \nu + 2$, $(V + \langle e_1, \ldots, e_\nu \rangle) \cap (V + \langle e_2, \ldots, e_{\nu+1} \rangle) = V$. So $F = V + x = V$. By Theorem 1.2 and all cases above,

$$\Omega_2 = \{ \langle e_2, \ldots, e_\nu \rangle \subseteq F \mid F \text{ is a maximal totally isotropic subspace} \}$$

is a clique containing both F_1 and F_2.

It is easy to check that there is no vertex of $\Omega_2 \setminus \Omega_1$ which is adjacent to all vertices of Ω_1. Hence Ω_1 is a maximal clique containing both F_1 and F_2. Similarly, Ω_2 ($\delta = 1$) is also a maximal clique containing both F_1 and F_2. Clearly, $|\Omega_1| = (q^{1/2} + 1)q$. By [5, Theorem 5.37], the number of maximal totally isotropic subspaces containing $\langle e_2, \ldots, e_\nu \rangle$ in the unitary space $\mathbb{F}_q^{(2\nu+1)}$ is equal to $q^{3/2} + 1$, i.e., $|\Omega_2| = q^{3/2} + 1$.

Secondly, suppose that F_1 and F_2 are as listed in (4). For any $F = V + x \in \Gamma(F_1) \cap \Gamma(F_2)$, we have $V = F_1$ and $F \neq F_1, F_2$ or $V \neq F_1$ and $x \in F_1 + V$ by Proposition 2.1. We distinguish the following two cases:

Case 1. $V = F_1$ and $F \neq F_1, F_2$. Then $x \notin F_1$ and $x - e_{\nu+1} \notin F_1$. So $F = F_1 + x$, where $x \in \langle e_{\nu+1}, \ldots, e_{2\nu+\delta}\rangle$, $x \neq 0$ and $x \neq e_{\nu+1}$. It follows from Theorem 1.2 that

$$\Omega_3 = \left\{\langle e_1, e_2, \ldots, e_\nu\rangle + x \;\middle|\; x \in \langle e_{\nu+1}, \ldots, e_{2\nu+\delta}\rangle\right\} = \left\{\langle e_1, e_2, \ldots, e_\nu\rangle + x \;\middle|\; x \in \mathbb{F}_q^{(2\nu+\delta)}\right\}$$

is a clique containing both F_1 and F_2.

Case 2. $V \neq F_1$ and $x \in F_1 + V$. Then $\dim(F_1 + V) = \nu + 1$ and $x - e_{\nu+1} \in F_1 + V$. So $e_{\nu+1} \in F_1 + V$. It follows that $F_1 + V = \langle e_1, e_2, \ldots, e_{\nu+1}\rangle$. Therefore, $F = V + x$, where $V \subseteq \langle e_1, e_2, \ldots, e_{\nu+1}\rangle$ is a maximal totally isotropic subspace and $V \neq F_1$, $x \in \langle e_1, e_2, \ldots, e_{\nu+1}\rangle$. It follows from Theorem 1.2 that

$$\Omega_1 = \left\{F \subseteq \langle e_1, e_2, \ldots, e_{\nu+1}\rangle \;\middle|\; F \text{ is a maximal totally isotropic flat}\right\}$$

is a clique containing both F_1 and F_2.

It is easy to check that there is no vertex of $\Omega_3 \setminus \Omega_1$ which is adjacent to all vertices of Ω_1. Hence Ω_1 is a maximal clique containing both F_1 and F_2. Similarly, Ω_3 is also a maximal clique containing both F_1 and F_2. Clearly, $\Omega_1 \neq \Omega_3$ and $|\Omega_3| = q^{\nu+\delta}$.

Finally, suppose that F_1 and F_2 are as listed in (5). Then $\delta = 1$. For any $F = V + x \in \Gamma(F_1) \cap \Gamma(F_2)$, we have $V = F_1$ and $F \neq F_1, F_2$ or $V \neq F_1$ and $x \in F_1 + V$ by Proposition 2.1. We distinguish the following two cases:

Case 1. $V = F_1$ and $F \neq F_1, F_2$. Then $x \notin F_1$ and $x - \lambda e_{2\nu+1} \notin F_1$. So $F = F_1 + x$, where $x \in \langle e_{\nu+1}, \ldots, e_{2\nu+1}\rangle$, $x \neq 0$ and $x \neq \lambda e_{2\nu+1}$. It follows from Theorem 1.2 that

$$\Omega_3 = \left\{\langle e_1, e_2, \ldots, e_\nu\rangle + x \;\middle|\; x \in \langle e_{\nu+1}, \ldots, e_{2\nu+1}\rangle\right\} = \left\{\langle e_1, e_2, \ldots, e_\nu\rangle + x \;\middle|\; x \in \mathbb{F}_q^{(2\nu+1)}\right\}$$

is a clique containing both F_1 and F_2.

Case 2. $V \neq F_1$ and $x \in F_1 + V$. Then $\dim(F_1 + V) = \nu + 1$ and $x - \lambda e_{2\nu+1} \in F_1 + V$. So $e_{2\nu+1} \in F_1 + V$. It follows that $F_1 + V = \langle e_1, e_2, \ldots, e_\nu, e_{2\nu+1}\rangle$. Therefore, $V \subseteq \langle e_1, e_2, \ldots, e_\nu, e_{2\nu+1}\rangle$ is a maximal totally isotropic subspace and $V \neq F_1$, a contradiction.

Since Ω_1, Ω_2 $(\delta = 1)$ and Ω_3 are all maximal cliques containing F_1, F_2, any maximal clique in Γ can be changed under the group $AU_{2\nu+\delta}(\mathbb{F}_q)$ into Ω_1, Ω_2 $(\delta = 1)$ or Ω_3, simultaneously. □

Proof of Theorem 1.5. For any $F \in V(\Gamma)$ and a given $(\nu + 1, 2)$-flat F' containing F, all maximal totally isotropic flats contained in F' form a maximal clique containing F by Theorem 1.4 and Proposition 2.1. By Proposition 2.1, the number of maximal cliques containing F is equal to the number of subspaces of type $(\nu + 1, 2)$ containing a given maximal totally isotropic subspace in the unitary space $\mathbb{F}_q^{(2\nu+\delta)}$. So by [5, Theorem 5.37], we can obtain that there exist $q^\delta(q^\nu - 1)/(q - 1)$ maximal cliques with size $q(q^{1/2} + 1)$ and containing F. From the proof of Theorem 1.4, we know that $\{F + x \mid x \in \mathbb{F}_q^{(2\nu+\delta)}\}$ is the only maximal clique with size $q^{\nu+\delta}$ and containing F. For $\delta = 1$ and a given $(\nu - 1)$-flat, denoted by F^*, contained in F, from the proof of Theorem 1.4, we know that $\{F' \mid F' \text{ is a maximal isotropic flat and } F^* \subseteq F'\}$ is the only maximal clique with size $q^{3/2} + 1$ and containing F. Clearly, the number of $(\nu - 1)$-flats contained in F is equal to $q(q^\nu - 1)/(q - 1)$.

Hence there are precisely $q^\delta(q^\nu - 1)/(q - 1) + \delta q(q^\nu - 1)/(q - 1) + 1$ maximal cliques containing F in Γ.

In order to compute the number of maximal cliques in Γ, we define M to be a binary matrix with row-indexed (resp. column-indexed) by $V(\Gamma)$ (resp. maximal cliques with size $q(q^{1/2} + 1)$), whose (A, B) entry $M(A, B) = 1$ if $A \in B$, and 0 otherwise. Counting the number of 1's in the matrix by rows, we obtain

$$\frac{|V(\Gamma)| q^\delta (q^\nu - 1)}{q - 1}.$$

Counting the number of 1's in the matrix by columns, we obtain

$$q\big(q^{1/2}+1\big)\alpha,$$

where α is the number of maximal cliques with size $q(q^{1/2}+1)$. Therefore,

$$\alpha=\frac{|V(\Gamma)|q^{\delta}(q^{\nu}-1)}{(q-1)q(q^{1/2}+1)}=\frac{q^{\nu-1+2\delta}(q^{\nu}-1)\prod_{i=1}^{\nu}(q^{i+\delta-1/2}+1)}{(q-1)(q^{1/2}+1)}.$$

Similarly, the number of maximal cliques with size $q^{\nu+\delta}$ is equal to $\prod_{i=1}^{\nu}(q^{i+\delta-1/2}+1)$, and the number of maximal cliques with size $q^{3/2}+1$ ($\delta=1$) is equal to

$$\frac{q^{\nu+2}(q^{\nu}-1)\prod_{i=2}^{\nu}(q^{i+1/2}+1)}{(q-1)}.$$

Hence the number of maximal cliques in Γ is equal to

$$\frac{q^{\nu-1+2\delta}(q^{\nu}-1)\prod_{i=1}^{\nu}(q^{i+\delta-1/2}+1)}{(q-1)(q^{1/2}+1)}+\prod_{i=1}^{\nu}\big(q^{i+\delta-1/2}+1\big)+\delta\frac{q^{\nu+2}(q^{\nu}-1)\prod_{i=2}^{\nu}(q^{i+1/2}+1)}{(q-1)}. \qquad \square$$

In subsequent papers we shall discuss the similar problems of affine-orthogonal and affine-pseudo-symplectic spaces.

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